

# A smooth codimension-one foliation of the five-sphere by symplectic leaves

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We construct a smooth codimension-one foliation on the five-sphere in which every leaf is a symplectic four-manifold and such that the symplectic structure varies smoothly. Our construction implies the existence of a complete regular Poisson structure on the five-sphere.

*Dedicated to the memory of James Eells (1926-2007).*

## 1 Introduction

In 1969, in a landmark paper in the theory of foliations, Blaine Lawson discovered the first example of a smooth codimension-one foliation of the sphere  $\mathbb{S}^5$  [8]. This example played a fundamental role in the development of the theory of foliations. In Lawson's foliation each leaf can be made *separately* into either a complex surface or a symplectic 4-manifold. In [12] Laurent Meersseman and the second author constructed a smooth codimension-one foliation of the 5-sphere by complex surfaces.

In this paper we show:

**Theorem 1.1** *There exists on  $\mathbb{S}^5$  a smooth codimension-one foliation by symplectic leaves.*

Whenever a smooth manifold admits a foliation by symplectic leaves, there exists an associated Poisson structure. Let us recall the definition and explain this point briefly. A Poisson structure on a manifold is a Poisson structure on its sheaf of real functions. This means that given two local functions  $f, g$  on the manifold we may define  $\{f, g\}$  on the overlap of their definition domains such that:

- $\{f, g\} = -\{g, f\}$

- $\{f, hg\} = g\{f, h\} + \{f, g\}h$
- $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

The Poisson bracket  $\{f, g\}$  can be given in terms of a Poisson tensor  $\Lambda$  by  $\Lambda(df, dg) = \{f, g\}$ . For a regular Poisson structure the tensor  $\Lambda$  defines an involutive distribution whose integral leaves come with a canonical symplectic structure. Conversely, any foliation by symplectic leaves such that for any smooth function the hamiltonians of the restriction of the function to each leaf glue into a smooth vector field induces a unique Poisson structure (the book by I. Vaisman [18] is a good reference to study the geometry of Poisson manifolds).

Therefore our result implies:

**Corollary 1.2** *The sphere  $\mathbb{S}^5$  admits a complete regular Poisson structure.*

**Remarks** (1) Notice that  $\mathbb{S}^5$  does not admit a plane field of dimension 2, since the Grassmannian  $G_{5,2}(\mathbb{S}^5) \rightarrow \mathbb{S}^5$  does not admit a section (see [3, p.10] and [15]). So that a Poisson structure for  $\mathbb{S}^5$  for which the associated symplectic foliation has 2-dimensional leaves can not exist either.

(2) We also wish to point out that  $\mathbb{S}^4$  has no regular nontrivial Poisson structures. As  $\mathbb{S}^4$  does not admit a symplectic structure, it can not admit a Poisson structure of rank 4. There is no structure of rank 2 because, if it were to exist, the orientations of its associated symplectic foliation and that of its transverse foliation would join to give  $\mathbb{S}^4$  an almost complex structure (which we know does not exist). This also follows since  $\mathbb{S}^4$  does not admit any 2-plane field.

(3) Any manifold  $X$  with an oriented smooth foliation by surfaces admits a regular Poisson structure of rank 2, by choosing an induced area form on each leaf associated to an arbitrary Riemannian metric on  $X$ .

(4) It has been shown by A Ibort and D Martínez-Torres that every finitely presented group can be realized as the fundamental group of a closed Poisson 5-manifold [7]. However, their surgery techniques do not apply to the five-sphere

(5) The construction in this paper is a positive answer to a question of Yasha Eliashberg to the second author.

Since  $\mathbb{S}^7$  is parallelizable, it has foliations of codimension 1 and 3, but it is not known if such a foliation can be made symplectic (in the sense that it induces a Poisson structure). It does have a foliation by surfaces so it has a Poisson structure of rank 2

[16, 17]. It appears that for homotopy spheres of higher dimension existence of such symplectic foliations is a hard problem and nothing else seems to be known.

In the following section we recall some standard definitions and we show how to glue foliations by symplectic leaves. We review in section 3 the necessary details from the construction of a smooth codimension-one foliation of  $\mathbb{S}^5$  by complex surfaces which was carried out by L Meersseman and the second author in [12]. In section 4 we construct symplectic forms adapted to the foliation of  $\mathbb{S}^5$  by complex surfaces and hence prove Theorem 1.1.

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## 2 Preliminaries

### 2.1 Definition of a symplectic form adapted to a foliation.

Let  $X$  be a smooth manifold.

**Definition 2.1** Let  $\mathcal{F}$  be a smooth foliation on  $X$  and write  $T\mathcal{F}$  for the tangent bundle of  $\mathcal{F}$ . A symplectic form associated to  $\mathcal{F}$  is a 2-form  $\omega$  on  $\Lambda^2\mathcal{F}$  which is closed and non-degenerate on each leaf of  $\mathcal{F}$  and which is  $C^\infty$  with respect to the foliation structure. See, for instance, [6].

Let  $X$  be a smooth manifold with boundary and  $Y$  be the open manifold obtained by adding the collar  $\partial X \times [0, 1)$  to  $X$ . The points  $x \in \partial X$  are identified with the points  $(x, 0) \in X \times [0, 1)$  and  $Y$  has a unique smooth structure such that the natural inclusions of  $X$  and  $\partial X \times [0, 1)$  into  $Y$  are smooth embeddings.

In analogy to [12, Definition 1] we will say a (codimension-one) *tame* symplectic foliation on  $X$  is the data of a symplectic foliation on the interior of  $X$  (in the previous sense) and of a symplectic structure on the boundary  $\partial X$  such that the following gluing condition is verified.

Let  $X \subset Y$  and  $\partial X \times [0, 1) \subset Y$  be the natural embeddings. Then the symplectic foliation on  $X$  extends to a smooth symplectic foliation on  $Y$  by considering on the collar the distribution which is tangent to the submanifolds  $\partial X \times \{t\}$ , for  $0 \leq t < 1$  and equipping this distribution with the natural symplectic structure inherited from  $\partial X$  (i.e.

the symplectic structure in  $\partial X \times \{t\}$  is given by  $\omega_t := \pi_t^*(\omega)$  where  $\pi_t : \partial X \times \{t\} \rightarrow \partial X$  is the natural projection).

The tameness condition allows for foliations to be glued together.

**Lemma 2.2** *Let  $X$  and  $X'$  be two manifolds with diffeomorphic non-vacuous boundary. Assume that there is a tame symplectic foliation on both of them such that their boundaries are symplectomorphic and such that the boundaries are leaves of the foliation. Let  $Y$  be the manifold obtained by gluing  $X$  and  $X'$  along their boundary by a symplectomorphism. Then  $Y$  admits a symplectic foliation which induces the original symplectic foliations on  $X$  and  $X'$ .*

For completeness sake we include a sketch of the proof, which is analogous to the proof of Lemma 1 in [12].

**Proof** The manifold  $Y$  has a unique  $C^\infty$  structure such that the natural inclusions  $X \subset Y$ ,  $X' \subset Y$  and  $\partial X \cong \partial X' \subset Y$  (where the identification is made through the given symplectomorphism) are smooth embeddings. The manifold  $Y$  has thus a continuous symplectic foliated structure  $\omega$ , which is smooth outside the submanifold  $\partial X \cong \partial X'$  and induces the original symplectic structure on  $X$  and  $X'$ .

We will now see that the tameness condition (see [12]) implies that the symplectomorphism which identifies the boundary components makes the symplectic foliation on  $X$  and  $X'$  extend to a smooth symplectic foliation on the whole of  $Y$ .

In fact under the gluing condition, this structure is smooth on all of  $Y$ . Take a point  $x \in \partial X \cong \partial X' \subset Y$  and let  $\{(x, t)\} := \{(x_1, \dots, x_{2n}, t)\}$  be local coordinates in  $X$  (modelled in all of  $\mathbb{R}^{2n} \times \mathbb{R}$ ) adapted to the submanifold  $\partial X \cong \partial X' \subset Y$ , (i.e. such that this submanifold is given locally by  $t = 0$ ). We assume that points with negative  $t$  lie in  $X$  whereas points with positive  $t$  lie in  $X'$  and the plaques of the foliation are given by  $t = \text{constant}$ . Then, the tameness condition implies that the symplectic structure  $\omega$  is given in these local coordinates as a smooth 2-form  $(x, t) \mapsto \omega(x, t)$  of the leaf  $\mathbb{R}^{2n}$  (at level  $t$ ) which is differentiable for  $t \geq 0$  and  $t \leq 0$  and is induced by the original structure on  $X$  for  $t \geq 0$  (denoted by  $\omega_+$ ) and by the original structure on  $X'$  for  $t \leq 0$  (denoted by  $\omega_-$ ). In fact the tameness condition plus the foliated version of Darboux theorem for regular Poisson manifolds ([19] Theorem 2.1) implies that there are local coordinates such that the form  $\omega(x, t)$  can be taken to be independent of  $t \in \mathbb{R}$  and equal to the standard symplectic form of  $\mathbb{R}^{2n}$ .  $\square$

We could also have modified  $\omega_+$  and  $\omega_-$  with an isotopy between them, by adding a collar to interpolate between them. In this situation we also appeal to tameness, because the added collar is a smooth foliation (trivial with respect to the collar coordinates). This construction adds an entire package of compact leaves to the foliations we began with, but also constructs a smooth path of symplectic forms from  $\omega_+$  to  $\omega_-$ .

This is done as follows:

Let  $V = \partial X \times [0, 1]$  be such a collar, recall that there exists a unique smooth structure for which all the natural inclusions are smooth embeddings. By Moser's Lemma there exists a smooth path  $\alpha(t)$  (from  $[0, 1]$ ) of symplectic structures  $\omega_{\alpha(t)}$  on the boundary  $\partial X \cong \partial X'$  such that  $\omega_{\alpha(t)} = \omega_+$  for  $0 \leq t \leq 1/4$  and  $\omega_{\alpha(t)} = \omega_-$  for  $3/4 \leq t \leq 1$ . On the collar  $\partial X \times [0, 1]$  at each level  $t \in [0, 1]$  we define the form  $\Omega_{\alpha(t)} := \pi_t^*(\omega_{\alpha(t)})$  where  $\pi_t : \partial X \times \{t\} \rightarrow \partial X$  is again natural projection.

The gluing condition implies that the forms  $\omega_+, \omega_-$  and  $\Omega_{\alpha(t)}$  define a symplectic structure adapted to the foliation on  $Y \cong X \cup_{f_+} V \cup_{f_-} X'$ . Here  $f_+$  and  $f_-$  are the maps which attach the collar  $V$  to  $\partial X$  and  $\partial X'$ , respectively.

A consequence is:

**Corollary 2.3** *Let  $Y$  be a domain with smooth boundary in a compact manifold  $X$ . Denote by  $\text{Int}(Y)$  the interior of  $Y$ . If both  $Y$  and  $X - \text{Int}(Y)$  have foliations by symplectic manifolds with symplectomorphic boundary leaves, then  $X$  admits a smooth foliation by symplectic manifolds.*  $\square$

### 3 Smooth foliations of $\mathbb{S}^5$ with complex leaves

The smooth foliations of  $\mathbb{S}^5$  with complex leaves constructed in [12] are obtained by decomposing  $\mathbb{S}^5$  as a union of two compact manifolds of dimension five which fiber over the circle and meet at their common boundary. This decomposition is obtained from an open book structure on  $\mathbb{S}^5$ . These manifolds with boundary have infinite cyclic coverings [9] with tame codimension one foliations whose leaves are complex surfaces. These foliations descend to the compact pieces and the common boundary is complex surface biholomorphic to a complex nilmanifold (a primary Kodaira surface).

We content ourselves with describing the two coverings and their tame foliations, referring to [12] for more details.

Consider the following polynomial

$$P: (z_1, z_2, z_3) \in \mathbb{C}^3 \longmapsto z_1^3 + z_2^3 + z_3^3 \in \mathbb{C}.$$

The manifold

$$W := P^{-1}(0) - \{(0, 0, 0)\} = \{(z_1, z_2, z_3) \neq 0 \mid z_1^3 + z_2^3 + z_3^3 = 0\}$$

intersects the Euclidean unit sphere transversally in the smooth compact manifold  $K$ . Notice that the complex tangencies to the Euclidean unit sphere induce a contact structure on the link  $K$ . In fact this induced contact structure is canonical and unique (see [1]).

The manifold  $W$  projects onto the projective space  $\mathbb{CP}^2$  as an elliptic curve  $\mathbb{E}_\omega$  of modulus  $\omega$ . This curve admits an automorphism of order three, hence  $\omega^3 = 1$ . The canonical projection

$$\pi: W \rightarrow \mathbb{E}_\omega$$

describes  $W$  as a holomorphic principal  $\mathbb{C}^*$ -bundle over the elliptic curve  $\mathbb{E}_\omega$ , with first Chern class equal to  $-3$ . By passing to the unit bundle, one has that  $K$  is a circle bundle over a torus with Euler class equal to  $-3$  (see [11] Lemma 7.1 and Lemma 7.2). More precisely, let the matrix  $A$  be given by

$$\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$$

then  $K$  is the suspension of the unipotent isomorphism induced on the two-dimensional torus by  $A$ . Let  $\mathcal{N}$  be a closed tubular neighborhood of  $K$  in  $\mathbb{S}^5$  identified, via a trivialization, with  $K \times D^2$  and  $\mathring{\mathcal{N}}$  its interior. The fiber bundle  $\pi$  has structure group  $\mathbb{C}^*$  and therefore we can reduce this group to the compact subgroup  $\mathbb{S}^1$  consisting of unit complex numbers. Moreover  $W$  is biholomorphic to the quotient of  $\mathbb{C}^* \times \mathbb{C}^*$  by an action of  $\mathbb{Z}$  of the form:

$$(n, w_1, w_2) \in \mathbb{Z} \times \mathbb{C}^* \times \mathbb{C}^* \longmapsto (\exp(2i\pi\omega n) \cdot w_1, (\psi(w_1))^{-3n} \cdot w_2) \in \mathbb{C}^* \times \mathbb{C}^*$$

where  $\psi: \mathbb{C}^* \rightarrow \mathbb{C}^*$  is an automorphic factor, associated to the  $\mathbb{C}^*$ -bundle, which is given as a quotient of  $\theta$ -functions.

The set of points  $(w_1, w_2)$  such that  $|w_2| = 1$  is invariant under this action and descends under  $\pi$  to a manifold which is diffeomorphic to  $K$ .

**Remark** In all that follows let  $\lambda$  be a real number such that  $\lambda > 1$  and  $d: \mathbb{R} \rightarrow \mathbb{R}$  is a smooth diffeomorphism such that  $d(t) = t$  if  $t \leq 0$  and its derivative satisfies  $d'(t) > 1$  when  $t > 0$ .

The following function

$$z \in \mathbb{C}^3 \longmapsto \lambda \omega \cdot z \in \mathbb{C}^3$$

leaves  $W$  invariant. The group generated by this transformation acts properly and discontinuously on  $W$  and the quotient is a compact complex manifold diffeomorphic to  $K \times S^1 \simeq \partial \mathcal{N}$ . Let us call it  $S_\lambda$ . The surface  $S_\lambda$  is a primary Kodaira surface which fibers over the elliptic curve  $\mathbb{E}_\omega$  with fibre the elliptic curve which is the quotient of  $\mathbb{C}^*$  by the action of the group generated by the homothetic transformation  $z \mapsto \lambda z$ .

Let us recall how the Reeb foliation in the solid torus was defined in [12]. Let  $A$  be the set  $(\mathbb{C} \times [-1, \infty) \setminus I)$ , where  $I$  is the set of point of the form  $(0, s)$  such that  $s$  is in  $[-1, 0]$ . Consider the action of the group  $\Gamma_\lambda$ , spanned by  $(u, t) \longmapsto (\lambda \cdot u, d(t))$  on  $A$ . The quotient  $A/\Gamma_\lambda$  is an open solid torus. The foliation by complex Riemann surfaces of  $(\mathbb{C} \times [-1, \infty) \setminus I)$  whose leaves correspond to  $t = \text{constant}$  for  $t \in [-1, \infty)$  descends to the solid torus. The leaves corresponding to  $t \in [-1, 0]$  descend to elliptic curves which are biholomorphic to the quotient of  $\mathbb{C}^*$  by the group generated by the homothetic transformation  $z \mapsto \lambda z$ . The leaves corresponding to  $t > 0$  descend to leaves biholomorphic to  $\mathbb{C}$ .

**Lemma 3.1** *Let  $\lambda$  be a real number with  $\lambda > 1$ . Let  $A$  be the set  $(\mathbb{C} \times [-1, \infty) \setminus I)$ , where  $I$  is the set of point of the form  $(0, s)$  such that  $s$  is in  $[-1, 0]$ . Consider the action of the group  $\Gamma_\lambda$ , spanned by  $(u, t) \longmapsto (\lambda \cdot u, d(t))$ . The action of  $\mathbb{C}^*$  on  $A$  by  $(u, t) \mapsto (\mu \cdot u, t)$ ,  $\mu \in \mathbb{C}^*$ , descends to the quotient  $A/\Gamma_\lambda$  and there exists a Riemannian metric on  $A$  which is invariant under the action of  $\mathbb{C}^*$  and therefore this metric descends to the solid torus  $A/\Gamma_\lambda$ .*

**Proof** Notice that  $\mathbb{C}^*$  acts on  $A$  by  $\mu(u, t) = (\mu \cdot u, t)$  if  $\mu \in \mathbb{C}^*$ , and this commutes with every element of  $\Gamma_\lambda$ . So it defines an action on the quotient. Therefore we may choose a metric  $ds^2$  which is invariant under the action of the compact subgroup of complex numbers of modulus one.

When this metric is lifted to  $A$  it becomes a smooth, even, radial function for each  $t$ . Therefore it takes the form  $\varphi(|u|^2, t)|du|^2$ , for a smooth real function  $\varphi$ .  $\square$

**Remark** The metric constructed above can be chosen to be *tame* in the following sense: the region covered by  $\mathbb{C} \times [-1, 0)$  is diffeomorphic to the product of a torus times an interval and the metric on each torus leaf can be chosen to be independent of the parameter on the interval, ie  $\varphi(|u|^2, t_1)|du|^2 = \varphi(|u|^2, t_2)|du|^2$  for all  $t_1$  and  $t_2$  in  $[0, 1]$ . This tameness condition will permit us to glue the symplectic foliations together.

Let

$$\tilde{\mathcal{N}} = \mathbb{C}^* \times (\mathbb{C} \times [0, \infty) \setminus \{(0, 0)\})$$

and let  $\Gamma$  be the group generated by the commuting diffeomorphisms  $T$  and  $S$  defined as follows:

$$\begin{aligned} \forall (z, u, t) \in \tilde{\mathcal{N}}, \quad T(z, u, t) &= (z, \lambda\omega \cdot u, d(t)) \\ \text{and} \quad S(z, u, t) &= (\exp(2i\pi\omega) \cdot z, (\psi(z))^{-3} \cdot u, t) \end{aligned}$$

where  $d$  is the smooth diffeomorphism defined before and  $\psi$  is the automorphic factor of  $W$  as  $\mathbb{C}^*$ -bundle over  $\mathbb{E}_\omega$ .

Let  $\Gamma_T$  be the group spanned by  $T$ . It can be read from the formula above that  $\tilde{\mathcal{N}}/\Gamma_T$  is a solid torus bundle over  $\mathbb{C}^*$ , as  $T$  leaves the first factor invariant and on the third one it acts by the "damped homothetic transformation"  $d(t)$ . Since  $S$  commutes with  $T$ , the map  $S$  induces a map  $\hat{S}$  on  $\tilde{\mathcal{N}}/\Gamma_T$  which sends the solid torus based on  $z \in \mathbb{C}^*$  to the solid torus based on  $\exp(2i\pi\omega) \cdot z \in \mathbb{C}^*$ , that is to say, it sends a solid torus to another solid torus.

If instead of  $S$  we use

$$S'(z, u, t) = \left( \exp(2i\pi\omega) \cdot z, \frac{\psi(z)^{-3}}{|\psi(z)^{-3}|} \cdot u, t \right)$$

then  $T$  and  $S'$  both act proper and discontinuously on  $\tilde{\mathcal{N}}$ . The quotient is again diffeomorphic to  $\mathcal{N}$  because the maps  $\psi(z)^{-3}$  and  $\frac{\psi(z)^{-3}}{|\psi(z)^{-3}|}$  are homotopic as maps from  $\mathbb{C}^*$  to itself, since they induce the same homomorphism on the fundamental group  $\pi_1(\mathbb{C}^*) = \mathbb{Z}$ .

The form

$$(1) \quad \omega_0 = \frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2} \oplus \varphi(|u|^2, t) \frac{i}{2} (du \wedge d\bar{u})$$

defined on  $\tilde{\mathcal{N}}$  is a foliated symplectic form associated to the smooth four-dimensional foliation whose leaves are given by  $t$  equals a constant. A direct calculation shows that action of  $S'$  on  $\tilde{\mathcal{N}}$  preserves this form, therefore it descends to the quotient and gives  $\mathcal{N}$  a tame foliated symplectic form. Moreover, the action of  $S'$  induces an action on  $\tilde{\mathcal{N}}/\Gamma_T$  by isometries on the solid tori with respect to the metric that we found in lemma (3.1). So that the area form on each torus fiber described in lemma (3.1) is left invariant and descends to the quotient.

Let  $\mathcal{F}_i$  be the foliation whose leaves  $L_t$  are the level sets in

$$\tilde{\mathcal{N}} = \mathbb{C}^* \times (\mathbb{C} \times [0, \infty) \setminus \{(0, 0)\})$$



of the projection onto the third factor. These leaves are naturally complex manifolds biholomorphic to  $\mathbb{C}^* \times \mathbb{C}$  for  $t > 0$  and  $\mathbb{C}^* \times \mathbb{C}^*$  for  $t = 0$ . The group  $\Gamma'$ , spanned by  $T$  and  $S'$  preserves the foliation  $\mathcal{F}_i$ , but it no longer acts biholomorphically on the leaves. However, we will show that it does preserve a symplectic form. It was shown in [12] that the quotient of  $\tilde{\mathcal{N}}$  by  $\Gamma$  is diffeomorphic to  $\mathcal{N}$  and thus provides a foliation whose leaves are diffeomorphic to complex manifolds on this closed set. Notice that  $\tilde{\mathcal{N}}/\Gamma$  and  $\tilde{\mathcal{N}}/\Gamma'$  are diffeomorphic, and in fact there exist diffeomorphisms which preserve the leaves. The boundary leaf  $K \times S^1$  is diffeomorphic to the complex primary Kodaira surface  $\mathcal{S}_\lambda$ , it is covered by  $L_{-1}$  and we will abuse our notation and still refer to it as  $L_{-1}$ . The other leaves are all diffeomorphic to complex line bundles over  $\mathbb{E}_\omega$  obtained from  $W$ —considered as  $\mathbb{C}^*$ —bundles over  $\mathbb{E}_\omega$ —by adding a zero section.

On the other hand, let  $g : \mathbb{C}^3 \times [-1, \infty) \rightarrow \mathbb{C}$  be the function defined by

$$g((z_1, z_2, z_3), t) = z_1^3 + z_2^3 + z_3^3 - \phi(t)$$

where  $\phi$  is a smooth function which is zero exactly on the non-positive real numbers.

Let  $\widehat{\Xi} = g^{-1}(\{0\})$  and  $\Xi = \widehat{\Xi} \setminus (\{(0, 0, 0)\} \times [-1, \infty))$ . Then,  $\Xi$  is a smooth manifold and has a natural smooth foliation  $\mathcal{F}_e$  by complex manifolds whose leaves  $\{L_t\}_{t \in [-1, \infty)}$  are parametrized by projection onto the factor  $[-1, \infty)$ .

Let  $G : \Xi \rightarrow \Xi$  be the diffeomorphism given by

$$(2) \quad G((z_1, z_2, z_3), t) = ((\lambda\omega \cdot z_1, \lambda\omega \cdot z_2, \lambda\omega \cdot z_3), h_\lambda(t))$$

where  $h_\lambda$  is a smooth diffeomorphism whose fixed points are 0 and  $(-\infty, -1]$ . For good choices of  $\phi$  and  $h_\lambda$  which are specified in [12], the pair  $(\Xi, \mathcal{F}_e)$  is a covering of the closure of  $\mathbb{S}^5 \setminus \mathcal{N}$  with deck transformation group  $\hat{\Gamma}$ , generated by  $G$ . The boundary is a leaf biholomorphic to  $\mathcal{S}_\lambda$  and the gluing condition is verified. There is another compact leaf corresponding to  $t = 0$ . It is also biholomorphic to  $\mathcal{S}_\lambda$ . These two compact leaves form the boundary of a collar whose interior leaves are all biholomorphic to  $W$ . Finally, the other leaves are all biholomorphic to the affine cubic surface  $P^{-1}(1)$  of  $\mathbb{C}^3$ . Notice that the action of  $G$  is holomorphic, unlike the action of  $\Gamma'$  defined above, and hence the foliation induced on the quotient is by complex manifolds.

**Remarks** (1) Observe that the second covering *is not* a product foliated covering since it has two topologically distinct leaves:  $W$  and  $P^{-1}(\{t\})$  for  $t \neq 0$ . But it is a union of product foliated coverings. Indeed, the restriction of  $\Xi$  to  $[-1, 0]$  is a product foliated covering, as well as its restriction to  $(-1, 0]$  and to  $(0, \infty)$ .

(2) The construction recalled above depends on the choices of the smooth functions  $d$ ,  $\phi$  and  $h_\lambda$ . Notice that  $d$  and  $h_\lambda$  define the holonomy of the compact leaves. As a consequence, if we construct such a foliation  $\mathcal{F}$  from  $d$  and  $h_\lambda$  and another one, say  $\mathcal{F}'$ , from  $d'$  and  $h'_\lambda$  with the property that  $d'$  (or respectively  $h'_\lambda$ ) is not smoothly conjugated to  $d$  (or respectively to  $h_\lambda$ ), then  $\mathcal{F}$  and  $\mathcal{F}'$  are not smoothly isomorphic, although they are topologically isomorphic (such maps exist, see [14]).

Nevertheless, the smooth type of the foliation is independent of the choice of the parameter  $\lambda$  in the following sense. Fix some  $\lambda$  and some smooth functions  $d$  and  $h_\lambda$  and call  $\mathcal{F}_\lambda$  the resulting foliation. Choose now a real number  $\mu$  with  $1 < \mu$  and different from  $\lambda$ . The function  $h_\lambda$  has the property that it coincides with the parabolic Möbius transformation  $t/(1 - 3(\log \lambda)t)$  near 0 (see [12, p. 925]). There exists a smooth function  $f: \mathbb{R} \rightarrow \mathbb{R}$  fixing 0 with the property that  $f \circ h_\lambda \circ f^{-1}$  coincides with the parabolic Möbius transformation  $t/(1 - 3(\log \mu)t)$ . It is easy to check that this new diffeomorphism can be used as  $h_\mu$ . As  $h_\mu$  is globally conjugated to  $h_\lambda$ , the foliation  $\mathcal{F}_\mu$  obtained from the previous construction using the functions  $d$  and  $h_\mu$  is smoothly isomorphic to  $\mathcal{F}_\lambda$ .

In what follows, we still talk of *the* foliation of [12], since the results we prove are independent of the particular choices of the functions  $d$  and  $h_\lambda$ . It will be important however to keep in mind the independence of the foliation with respect to  $\lambda$ , as was indicated above.

## 4 Symplectic forms adapted to the foliation of $\mathbb{S}^5$ by complex surfaces

We will adapt the ideas of the previous section to a notation which will conveniently describe the symplectic foliation. Remember that we have two compact leaves, one of which we will call the *interior* leaf and the other will be called the *exterior* leaf, as explained below.

### 4.1 A symplectic structure adapted to a neighbourhood of the interior compact leaf of the complex foliation of $\mathbb{S}^5$

Let  $\mathbb{C}$  denote the complex plane. Consider the space:

$$\tilde{X} := \mathbb{C}^* \times (\mathbb{C} \times [-1, \infty) - \{(0, 0)\})$$

On  $\mathbb{C}^*$  consider the symplectic form  $\frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2}$ , and on  $\mathbb{C}$  the standard symplectic form  $\frac{i}{2}(du \wedge d\bar{u})$ .

For global coordinates  $(z, u, t)$  on  $\tilde{X}$  the form (1):

$$\frac{i}{2} \frac{dz \wedge d\bar{z}}{|z|^2} \oplus \varphi(|u|^2, t) \frac{i}{2}(du \wedge d\bar{u})$$

is a foliated symplectic form on  $\tilde{X}$ .

As it was explained in section (3) that for all  $(z, u, t) \in \tilde{X}$  the commuting diffeomorphisms

$$T(z, u, t) = (z, \lambda\omega u, d(t))$$

$$\text{and } S'(z, u, t) = \left( \exp(2i\pi\omega) \cdot z, \frac{(\psi(z))^{-3}}{|(\psi(z))^{-3}|} \cdot u, t \right)$$

span a group  $\Gamma'$  which preserves the foliation whose leaves  $L_t$  are the level sets in  $\tilde{X}$  of the projection to the third factor. Moreover, this foliation descends to a foliation by symplectic 4-manifolds on a manifold which we can describe in two steps. The quotient of  $\mathbb{C}^* \times (\mathbb{C} \times [0, \infty) - \{(0, 0)\}) \subset \tilde{X}$  by the action of  $\Gamma'$ , which we will denote by  $X$ , is diffeomorphic to  $K \times D^2$ . The compact leaf  $K \times S^1$  (which we will call the interior compact leaf) is precisely the quotient of the boundary of  $\mathbb{C}^* \times (\mathbb{C} \times [0, \infty) - \{(0, 0)\})$ .

The quotient of the region  $\mathbb{C}^* \times (\mathbb{C} \times [-1, 0] - \{(0, 0)\})$  by the action of  $\Gamma'$  is diffeomorphic to a one-parameter family of  $\mathbb{C}^*$ -bundles over an elliptic curve. The two boundary components here each correspond to the compact leaves of the foliation of  $\mathbb{S}^5$  by complex surfaces constructed in [12, p.923].

The Riemannian metric on the solid tori described in lemma (3.1) induces an area form on the leaves of the Reeb foliation on the interior of the solid tori. This area form is invariant under the action of  $S'$ , therefore it induces a symplectic form associated to the foliation on the quotient of  $\tilde{\mathcal{N}}/\Gamma_T$  under the action of the group spanned by  $S'$ . Hence this gives a symplectic structure associated to the foliation on the closed set  $X$  which is diffeomorphic to the closed tubular neighbourhood  $\mathcal{N} \cong K \times D^2$  of  $K$  in  $\mathbb{S}^5$ .

**Remark** From now on we will think of  $\mathcal{N}$  and  $X$  as if they were the same smooth compact manifolds with boundary (since they are diffeomorphic), thus we regard  $X$  as contained in  $\mathbb{S}^5$ . In particular we identify the leaf  $L_{-1}$  with the boundary of  $\mathcal{N}$ .

## 4.2 A symplectic structure adapted to a neighbourhood of the exterior compact leaf of the complex foliation of $\mathbb{S}^5$

In the notation of the previous section, denote by  $L_{-1}$  the exterior compact leaf of the foliation by complex surfaces of  $\mathbb{S}^5$ . As explained above  $L_{-1}$  bounds the set  $\mathcal{N}$ .

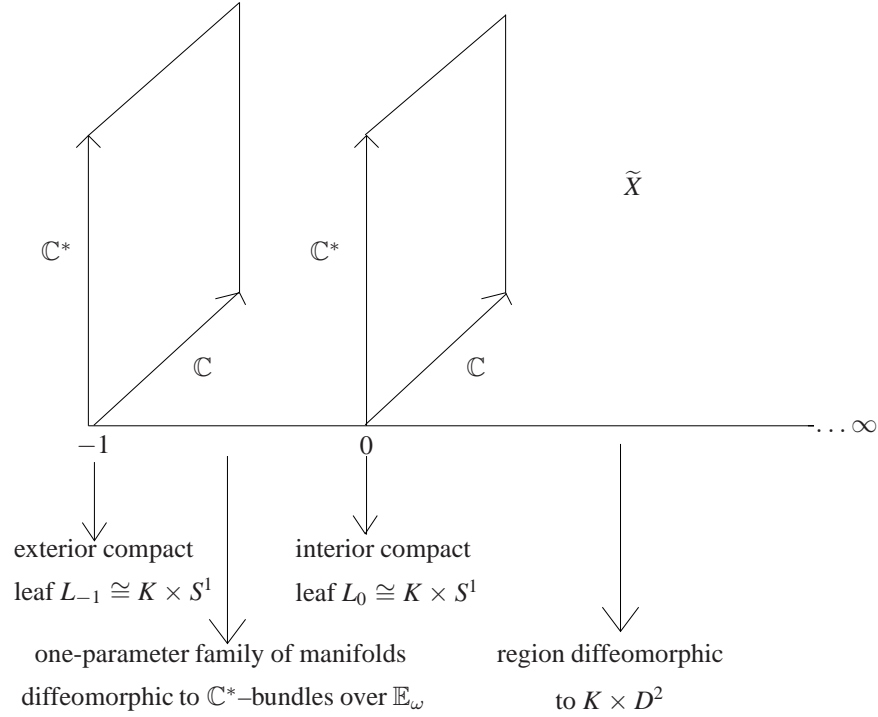


Figure 1: How the various regions of  $\tilde{X}$  cover parts of the foliation of  $\mathbb{S}^5$  by complex surfaces.

We will now describe a symplectic foliation in a neighbourhood  $C$  of  $\partial\mathcal{N} = L_{-1}$  in the complement  $\mathcal{N}^c$  of  $\mathcal{N}$  in  $\mathbb{S}^5$ . Recall that in a small product neighbourhood  $C$  of  $\partial\mathcal{N}$  in  $\mathcal{N}^c$  the foliation by complex surfaces of  $\mathbb{S}^5$  constructed in [12] is such that the leaves are asymptotic to the exterior compact leaf  $L_{-1} \cong K \times S^1$ . Notice that  $C$  has two boundary components; one of which is  $L_{-1}$  and we will denote the other one by  $\partial^+C$ . The leaves spiral around and accumulate to  $L_{-1}$  because the holonomy of the leaf  $L_{-1}$  is contracting. This is explained clearly in [13]. The fact that the holonomy of  $L_{-1}$  is contracting also implies that the leaves, different from  $L_{-1}$ , in  $C$  meet  $\partial^+C$  transversally in a 3-manifold which is diffeomorphic to  $K$ . The intersection of these leaves with  $\partial^+C$  foliate  $\partial^+C$  by copies of  $K$ . This foliation is equivalent to the product foliation  $K \times S^1$ .

We recall that the open set  $\mathcal{N}^c$  is foliated by leaves which are biholomorphic to the nonsingular affine Fermat cubics. If  $L$  is any of those leaves then the intersection of  $L$  with the closed neighborhood  $C$  is diffeomorphic to  $K \times [0, \infty)$ .

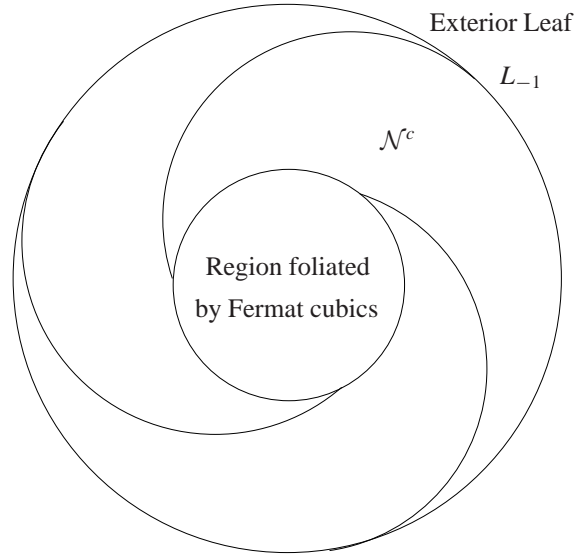


Figure 2: In a neighbourhood of the exterior leaf  $L_{-1}$  inside  $\mathcal{N}^c$ , the leaves are asymptotic to the exterior leaf and become transversal to the boundary of a domain in the Fermat region.

The foliated symplectic structure  $\omega_0$  of  $X = \mathcal{N}$  described in the previous section gives its boundary leaf  $L_{-1} \cong K \times S^1$  a symplectic structure, which is  $\omega_0|_{L_{-1}}$  and will be denoted by  $\omega_{-1}$ . Now we consider the projection map

$$p: C \rightarrow K \times S^1.$$

To each leaf  $L$  in  $C$  we assign the symplectic structure  $p^*\omega_{-1}$  by pulling back  $\omega_{-1}$  via the restriction of  $p$  to  $L$ .

We recall that every leaf in this region different from  $L_{-1}$  is diffeomorphic to  $K \times [0, \infty)$ . The pull-back  $p^*\omega_{-1}$  is symplectomorphic to  $d(e^s \alpha)$ , where  $\alpha$  is a contact form for the canonical contact structure on  $K$  as a Milnor fiber and  $s$  lies in  $[0, \infty)$ . Every leaf in this region is a symplectic manifold (with contact boundary) symplectomorphic to the manifold

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^3 + z_2^3 + z_3^3 = 0, |z_1|^2 + |z_2|^2 + |z_3|^2 > 1\}.$$

In fact  $C - L_{-1}$  is diffeomorphic to the noncompact manifold with boundary  $M_\delta$  defined, for  $\delta > 0$  sufficiently small, as follows:

$$M_\delta = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^3 + z_2^3 + z_3^3 = \delta e^{2\pi i \theta}, |z_1|^2 + |z_2|^2 + |z_3|^2 \geq 1, \theta \in [0, 2\pi)\}$$

The boundary of  $M_\delta$  is:

$$\partial(M_\delta) = \{(z_1, z_2, z_3) \mid z_1^3 + z_2^3 + z_3^3 = \delta e^{2\pi i \theta}, |z_1|^2 + |z_2|^2 + |z_3|^2 = 1, \theta \in [0, 2\pi)\}$$

One has, by the proof of Milnor's fibration theorem [10], that  $\partial(M_\delta) = K \times S^1$ .

We have obtained:

**Lemma 4.1** *The symplectic forms  $p^*\omega_{-1}$  are adapted to the foliation by complex surfaces of the neighbourhood  $C$  of the exterior leaf  $L_{-1}$ , and they constitute a foliated symplectic form. Furthermore every leaf different from  $L_{-1}$  is a symplectic manifold with contact boundary. Each of these boundaries are contactomorphic to  $K$  with the canonical contact structure induced as a Milnor fiber.*  $\square$

### 4.3 Symplectic structures on the region foliated by Fermat cubics

Before proceeding let us recall some facts about the Milnor fibration theorem. Milnor shows in [10] that if we define  $F_\theta$  to be the set of points  $(z_1, z_2, z_3) \in \mathbb{C}^3$  such that

$$z_1^3 + z_2^3 + z_3^3 = \delta \exp(2\pi i\theta)$$

and

$$|z_1|^2 + |z_2|^2 + |z_3|^2 \leq 1,$$

then

$$E_\delta = \bigcup_{\theta \in [0,1]} F_\theta$$

is isomorphic to the Milnor fibration in the complement of the interior of an open tubular neighborhood of the link (for  $\delta$  sufficiently small). This open tubular neighborhood of the link can be taken to be equal to the interior of  $\mathcal{N} \cup C$ .

For each  $\theta$ , the intersection of  $F_\theta$  with the unit sphere  $\mathbb{S}^5$  has a canonical contact structure and this is isomorphic to the one on the link  $K$  (see [1]). Obviously these contact structures admit a Stein filling, since the set of points such that

$$z_1^3 + z_2^3 + z_3^3 = \delta \exp(2\pi i\theta)$$

is a Stein surface. Therefore  $E_\delta$  has a smooth foliation by Stein surfaces with their canonical symplectic form inherited from  $\mathbb{C}^3$ , these meet the boundary of  $E_\delta$  in a (Stein fillable) contact manifold which is a copy of  $K$ . We will call this foliated symplectic form  $\omega_F$ .

From lemma (4.1) and the above discussion we obtain, as a summary, the following proposition:

**Proposition 4.2** *The compact submanifold with boundary  $\mathcal{N} \cup C \subset \mathbb{S}^5$  has a smooth foliation by symplectic leaves with foliated symplectic form  $\omega_F$ . Each leaf which intersects the boundary  $\partial(\mathcal{N} \cup C) = K \times S^1$  meets it transversally in  $K \times \{\theta\}$ , for some  $\theta \in S^1$ . Furthermore, for each  $\theta$ , the manifold  $K \times \{\theta\}$  inherits the canonical Milnor fillable contact structure  $\xi$  (which is independent of  $\theta$ ).  $\square$*

To end the proof of the theorem (1.1) we only need to glue  $E_\delta$  and  $\mathcal{N} \cup C$ . The boundaries of these manifolds are diffeomorphic to  $K \times S^1$  and a leaf that meets the boundary meets it in the contact manifold  $(K \times \{\theta\}, \xi)$ .

In order to do this we appeal to the following fact, which is a version of Gotay's symplectic neighborhood theorem [5] for products with  $S^1$  and whose omitted proof is a straightforward generalization of the results in [5] (see also [4, Remark 3.6, p.288]):

**Lemma 4.3** *Let  $M_1$  and  $M_2$  be two compact manifolds with diffeomorphic boundary components  $N_1$  and  $N_2$ . Let  $\omega_1$  and  $\omega_2$  be foliated symplectic forms on  $M_1$  and  $M_2$ , respectively. Suppose:*

- *$N_i$  is diffeomorphic to  $Q \times S^1$ , with  $Y$  a fixed contact manifold (for  $i \in \{1, 2\}$ ).*
- *There exist codimension-one foliations  $\mathcal{F}_1$  and  $\mathcal{F}_2$  by symplectic leaves (as in definition (2.1)) on  $M_1$  and  $M_2$ , respectively.*
- *The leaves of  $\mathcal{F}_i$  which meet the boundary  $N_i$  intersect it transversally in a copy of  $Q \cong Q \times \{\theta\} \subset Q \times S^1$ , for some  $\theta \in S^1$  (and  $i \in \{1, 2\}$ ).*
- *There exists an orientation-reversing diffeomorphism  $f: N_1 \rightarrow N_2$  such that  $f^*\omega_1 = \omega_2$  and is independent of the  $S^1$  coordinate of  $N_i \cong Q \times S^1$ .*

*Then the manifold  $M = M_1 \cup_f M_2$  admits a canonical, up to foliated Hamiltonian diffeomorphisms, foliated symplectic structure  $\omega$ .  $\square$*

We apply the previous lemma taking

$$M_1 = \mathcal{N} \cup C \subset \mathbb{S}^5$$

and

$$M_2 = E_\delta \cong \mathbb{S}^5 - \text{Int}(\mathcal{N} \cup C).$$

Then with  $Q = K$  we have  $N_1 \cong K \times S^1 \cong N_2$ . The symplectic structure  $\omega_1$  in  $M_1 = \mathcal{N} \cup C$  is the symplectic structure which is equal to  $\omega_0$  in  $\mathcal{N}$  and equal to  $p^*\omega_{-1}$  in  $C$ . In  $M_2 = E_\delta$  the symplectic form  $\omega_2$  equals the form  $\omega_F$  defined in proposition

(4.2). By the above, we have:  $(C - L_{-1}) \cong M_\delta$  and we can identify  $(C - L_{-1}) \cup E_\delta$  with

$$P^{-1}(S_\delta^1) = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^3 + z_2^3 + z_3^3 = \delta e^{2\pi i \theta}, \theta \in [0, 2\pi)\},$$

where  $S_\delta^1$  is the circle of radius  $\delta$  centered at the origin of  $\mathbb{C}$ . The orientation condition is automatically satisfied since  $C - L_{-1}$  is oriented as  $M_\delta$  and  $M_2$  is oriented as  $E_\delta$ .

This finishes the proof of Theorem 1.1.

In fact, in the course of the proof we have shown that  $K \times D^2$  and  $\mathbb{S}^5 - \text{Int}(\mathcal{N})$  each have smooth and tame symplectic foliations by 4-manifolds with common boundary  $K \times S^1$ .

Taking doubles of each of these manifolds we obtain by our gluing lemma (2.2):

- Corollary 4.4** (1) *The 5-manifold  $K \times S^2$  has a complete and regular Poisson structure of rank 4 whose associated non-singular and smooth symplectic foliation has three compact leaves, two generalized Reeb components and two open collars whose leaves are diffeomorphic to  $\mathbb{C}^*$ -bundles over an elliptic curve.*
- (2) *Let  $X$  be the double of  $\mathbb{S}^5 - \text{Int}(\mathcal{N})$ . Then  $X$  fibers over  $S^1$ , with fiber the double of the closed Milnor fiber which is defined by*

$$\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^3 + z_2^3 + z_3^3 = \eta \leq 1 \text{ and } |z_1|^2 + |z_2|^2 + |z_3|^2 \leq 1\}.$$

*Moreover,  $X$  admits a complete and regular Poisson structure of rank 4 whose associated non-singular and smooth symplectic foliation has one compact leaf and two generalized Reeb components.*

## Final Remarks

- (1) We could have taken  $K$  to be the link of the  $(2, 4, 4)$  singularity or the  $(2, 3, 6)$  singularity and our construction would carry through almost identically, in particular it would be a different construction for Theorem 1.1.
- (2) The foliated version of Moser's theorem [6] implies that the space of deformations of this symplectic foliation is always trivial. In contrast to the holomorphic foliation of  $\mathbb{S}^5$ , whose space of deformations is  $\mathbb{C}^3$  [13].

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